R_0 in a periodic environment

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R_0 and related concepts: methods and illustrations
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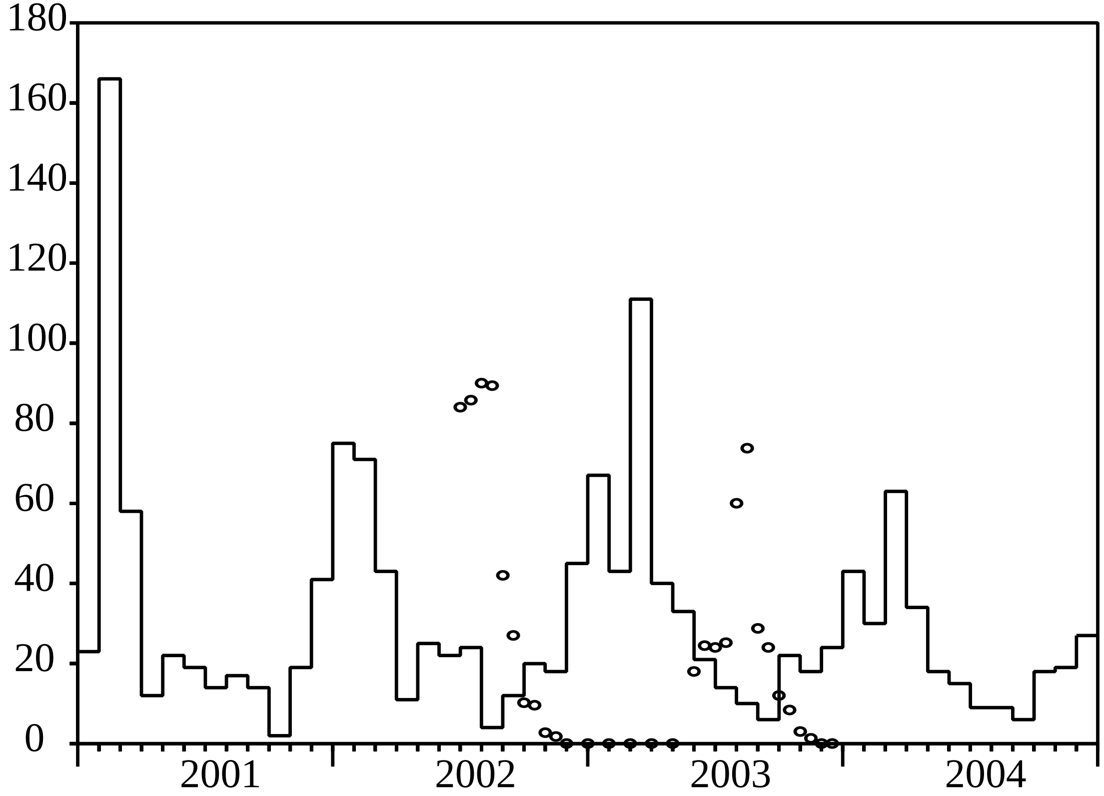
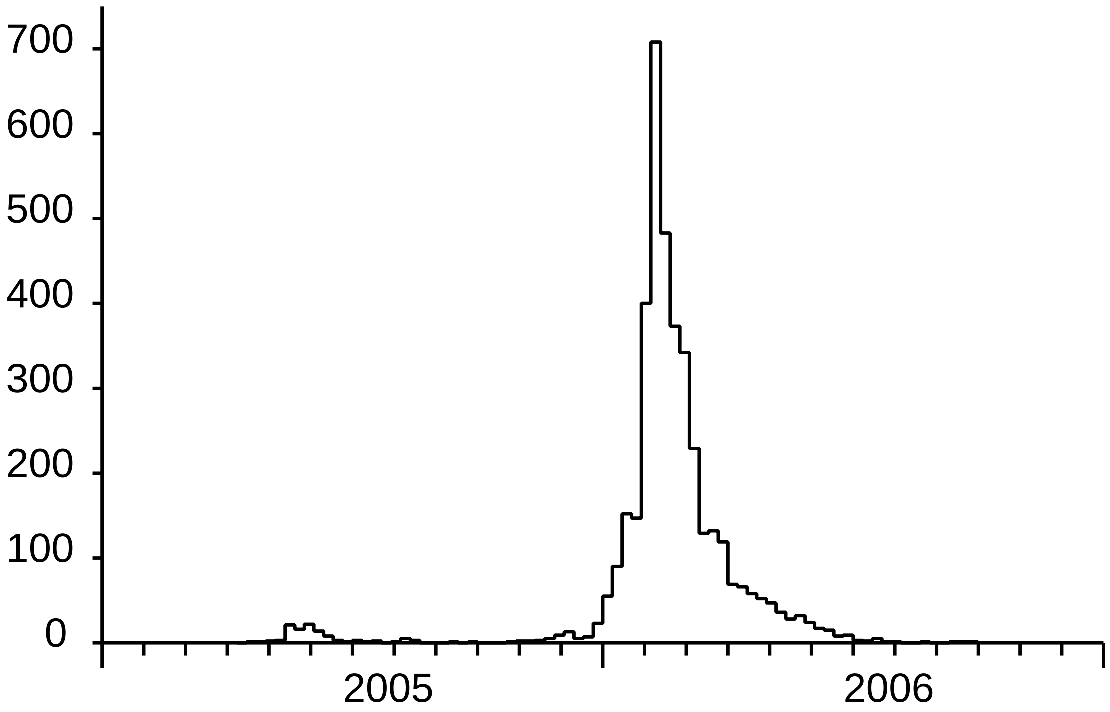
leishmaniasis in Morocco



chikungunya in La Réunion (Indian Ocean)



Discrete-time models

$$n(t) = \sum_{x=1}^{\infty} K(t,x)\, n(t-x)$$

$$K(t+\tau, x) = K(t,x)$$

$$R_0\, u(t) = \sum_{x=1}^{\infty} K(t,x)\, u(t-x)$$

If $\tau = 1$, R_0 is the spectral radius of $\sum_{x=1}^{\infty} K(x)$

Example

$$p(t+1) = (T(t) + F(t))\, p(t)$$

R_0 is the spectral radius of

$$\begin{pmatrix} F(0) & 0 & \cdots & 0 \\ 0 & F(1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & F(\tau-1) \end{pmatrix} \begin{pmatrix} -T(0) & I & 0 & \cdots & 0 \\ 0 & -T(1) & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & I \\ I & 0 & \cdots & 0 & -T(\tau-1) \end{pmatrix}^{-1}$$

If $\tau = 1$, R_0 is the spectral radius of $F(I-T)^{-1}$

Continuous-time models

$$n(t) = \int_0^\infty K(t,x)\, n(t-x)\, dx$$

$$K(t+\tau, x) = K(t,x)$$

$$R_0\, u(t) = \int_0^\infty K(t,x)\, u(t-x)\, dx$$

Example 1

$$\frac{dp}{dt} = a(t)\, p(t) - b(t)\, p(t)$$

$$R_0 = \frac{\int_0^\tau a(t)\, dt}{\int_0^\tau b(t)\, dt}$$

Example 2

$$\frac{dp}{dt} = A(t)\, p(t) - B(t)\, p(t)$$

R_0 is such that the dominant Floquet multiplier of

$$\frac{dp}{dt} = \frac{A(t)}{R_0}\, p(t) - B(t)\, p(t)$$

is equal to 1.

Example 3

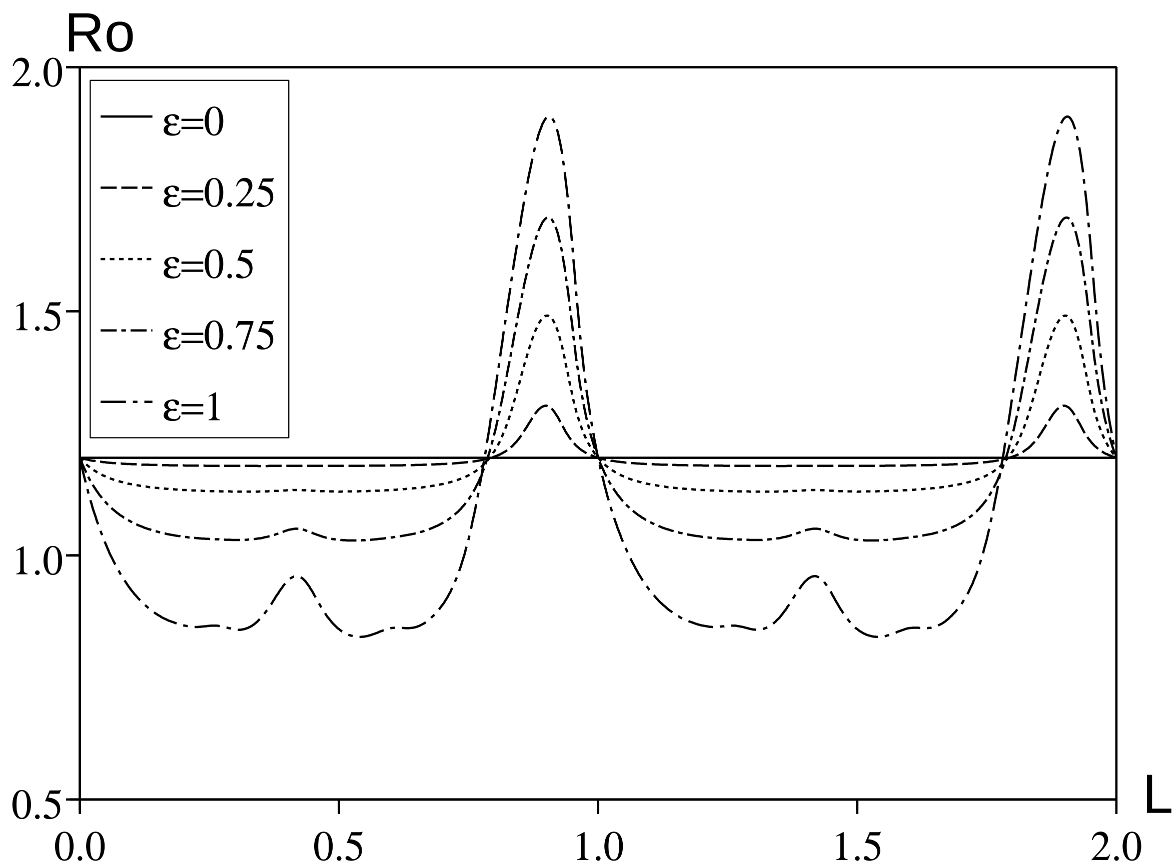
$$K(t,x) = (1+\varepsilon\cos\omega t)k(x)$$

$$\frac{R_0}{k_0} - 1 = 2\,\Re\ \cfrac{\varepsilon^2/4}{\cfrac{R_0}{k_1} - 1 - \cfrac{\varepsilon^2/4}{\cfrac{R_0}{k_2} - 1 - \cfrac{\varepsilon^2/4}{\cdots}}}$$

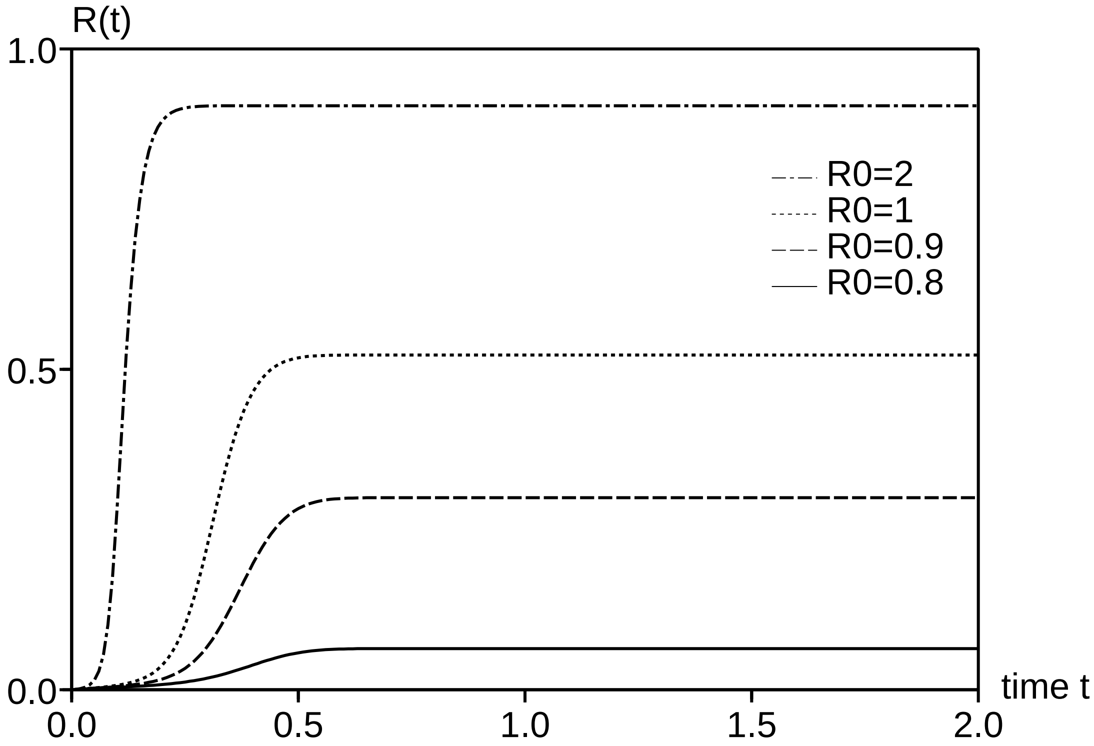
$$k_n = \int_0^\infty e^{-n\mathrm{i}\omega x}\,k(x)\,dx$$

$$R_0 \underset{\varepsilon\to 0}{\simeq} k_0 + \frac{\varepsilon^2}{2}\Re\left(\frac{k_0\,k_1}{k_0 - k_1}\right)$$

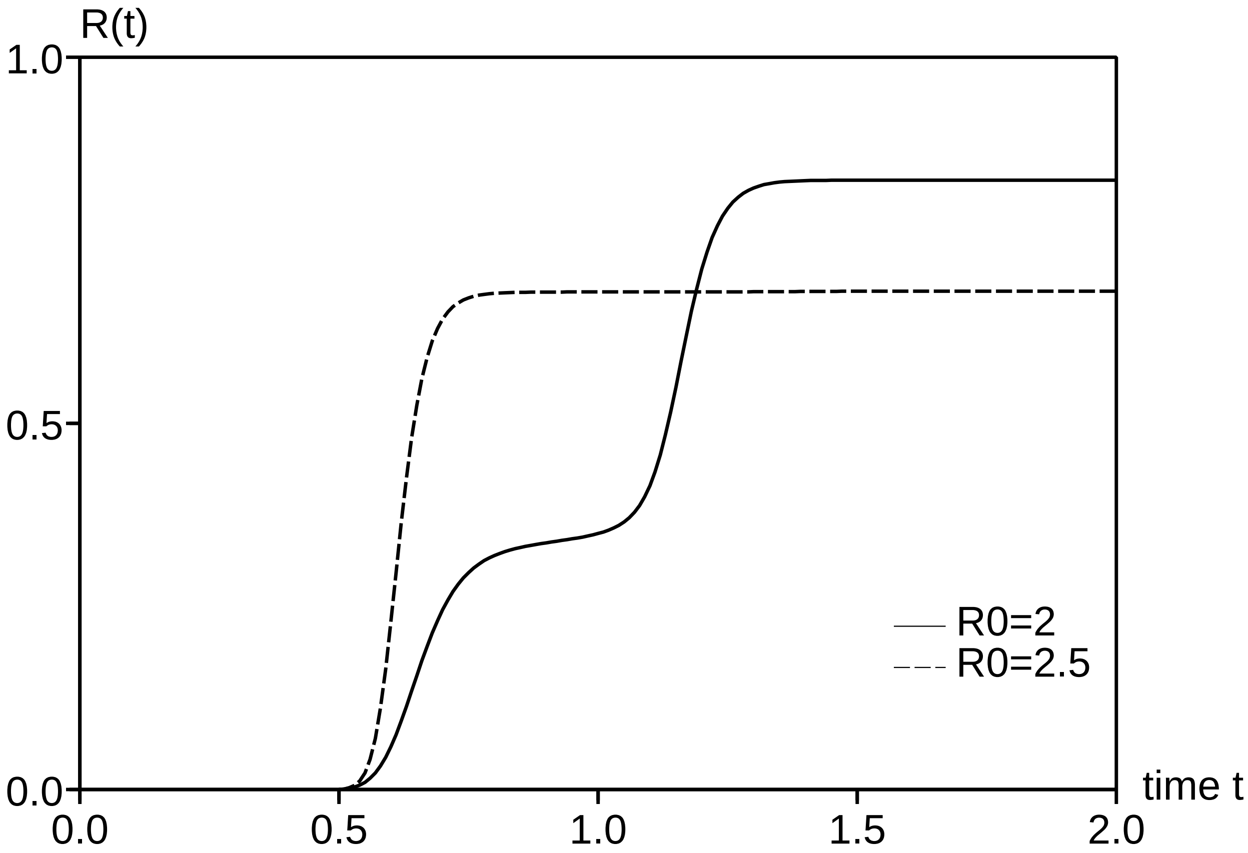
Warning 1: resonance of R_0 (SEIR)



Warning 2: large epidemics possible when $R_0 < 1$ and initial seed at the right time and not too small (SIR)

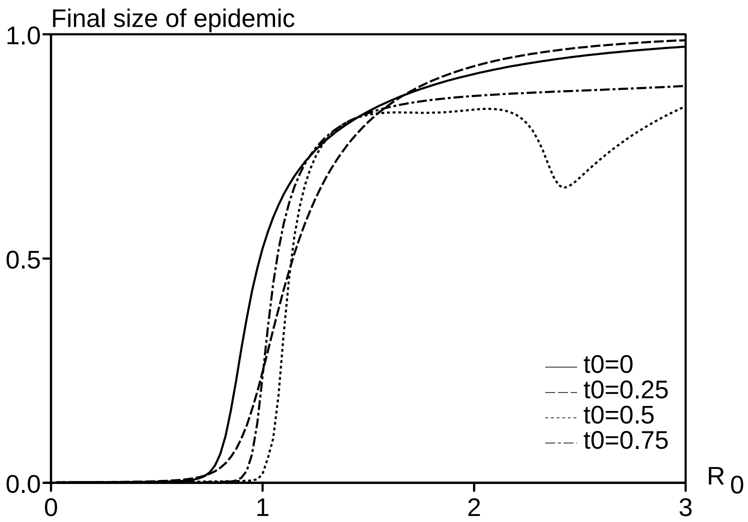


Warning 3: final size may not increase with R_0 (SIR)

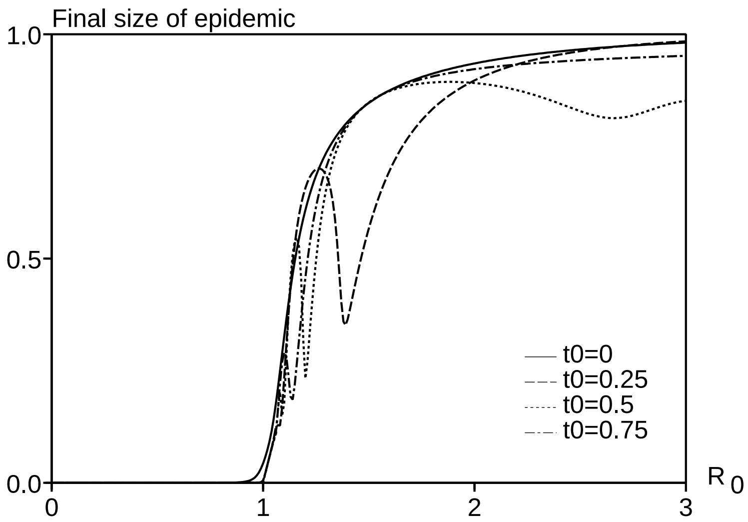


Warnings 2 and 3: summary (SIR)

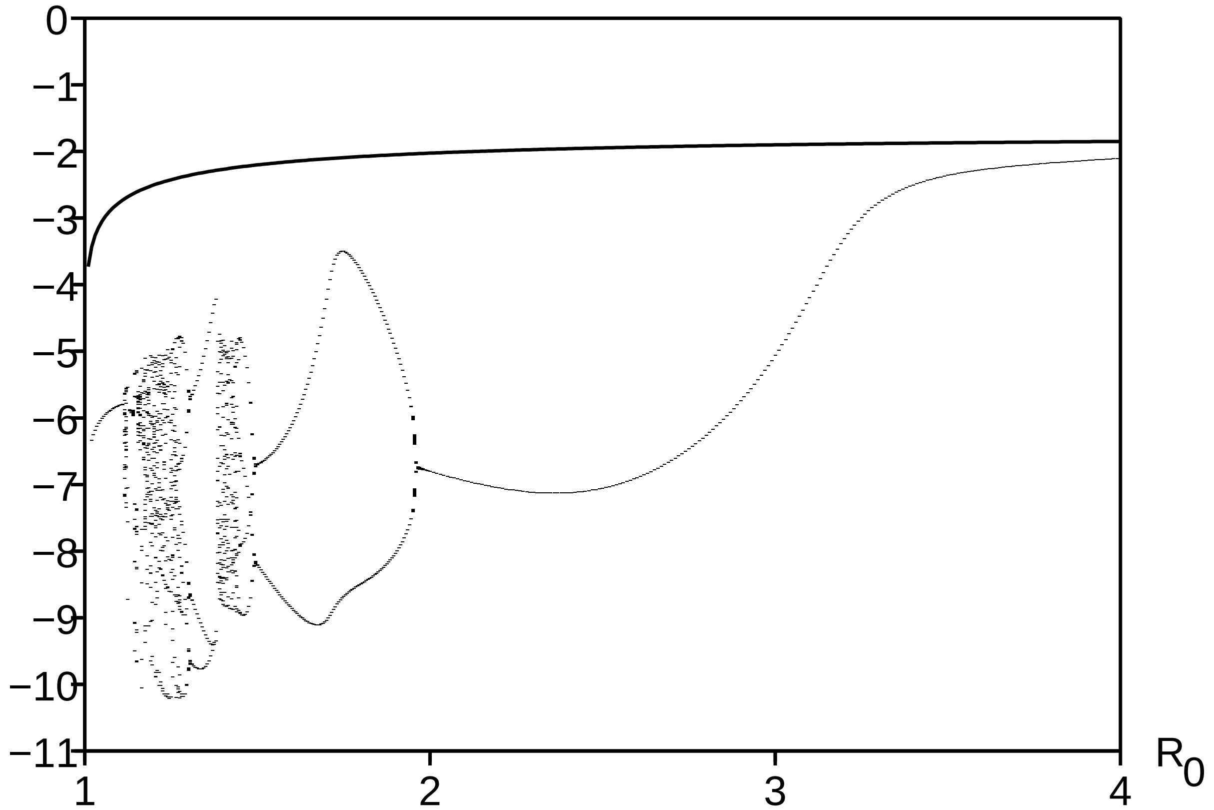
$I(t_0) = 10^{-3}$



$I(t_0) = 10^{-6}$



Warning 4: endemicity if $R_0 > 1$ can be unrealistic (SIRS)



Related concept: the Malthusian parameter

$$p(t+1) = A(t)\,p(t)$$

$$p(t) \underset{t\to\infty}{\simeq} c\,\lambda^t\,r(t)$$

λ is the spectral radius of

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & A(\tau-1) \\ A(0) & 0 & \cdots & 0 & 0 \\ 0 & A(1) & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & A(\tau-2) & 0 \end{pmatrix}$$

$(\ell(0), \ldots, \ell(\tau-1))$ left-eigenvector

$(r(0), \ldots, r(\tau-1))$ right-eigenvector

$$\frac{\partial \lambda}{\partial A_{i,j}(t)} = \frac{\ell_i(t+1)\, r_j(t)}{\tau\, \langle \ell(0), r(0) \rangle}$$

$$\langle \ell(t), p(t) \rangle = \lambda^t \langle \ell(0), p(0) \rangle$$

$$\langle \ell(t), r(t) \rangle = \langle \ell(0), r(0) \rangle$$

$$\pi_i(t) = \frac{\ell_i(t)\, p_i(t)}{\lambda^t \langle \ell(0), p(0) \rangle} \;, \qquad \omega_i(t) = \frac{\ell_i(t)\, r_i(t)}{\langle \ell(0), r(0) \rangle}$$

$$t \longmapsto \sum_{i=1}^{s_t} \pi_i(t) \log \frac{\pi_i(t)}{\omega_i(t)} \text{ is decreasing to } 0$$

References

- N.B., S. Guernaoui: The epidemic threshold of vector-borne diseases with seasonality - The case of cutaneous leishmaniasis in Chichaoua, Morocco. J. Math. Biol. (2006)

- N.B.: Approximation of the basic reproduction number R_0 for vector-borne diseases with a periodic vector population. Bull. Math. Biol. (2007)

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- N.B., X. Abdurahman: Resonance of the epidemic threshold in a periodic environment. J. Math. Biol. (2008)

- N.B.: A simple formula for the sensitivity analysis of periodic matrix population models. Submitted.

- N.B., M.G.M. Gomes: Some further notes on the basic reproduction number in a periodic environment. Submitted.

R_0 in models with heterogeneous contact rates

Distribution of the number of commercial sex partners for 407 adult males and 403 female sex workers in Yunnan (China)

	mean	standard deviation
adult males (last 12 months)	1.0	2.2
sex workers (last week)	3.0	4.1

Let $F(x)$ be the distribution of contact rates $x > 0$.
Anderson/May: $R_0 \sim \langle x^2 \rangle / M = M + V/M$
(M mean, V variance)

The probability of having k partners over a fixed time span t is a mixed Poisson distribution:

$$f_k(t) = \int_0^\infty e^{-xt} \frac{(xt)^k}{k!} F(x) dx$$

$$\Rightarrow m = t\, M\,, \quad v = t^2 V + m$$

$$\Rightarrow M + V/M = (m + v/m - 1)/t$$

N.B., X. Abdurahman, J. Ye, P. Auger: On the basic reproduction number R_0 in sexual activity models for HIV/AIDS epidemics: Example from Yunnan, China. Math. Biosci. Eng. (2007)